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Vortex identification: New requirements and limitations

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Abstract

Firstly, a brief survey dealing with popular vortex-identification methods is presented. The most widely used local criteria (applied point by point) – sharing a basis in the velocity-gradient tensor $\nabla \mathbf{u}$ – are treated more thoroughly to recall their underlying ideas and physical aspects. A large number of recent papers have pointed out various applicability limitations of these popular schemes and formulated (explicitly or implicitly) new general requirements, for example: validity for compressible flows and variable-density flows, determination of the local intensity of swirling motion, vortex-axis identification, non-local properties, ability to provide the same results in different rotating frames, etc. Other quite natural requirements are pointed out and added to those already mentioned. Secondly, the vortex-identification outcome of the proposed triple decomposition of the relative motion near a point is presented. The triple decomposition of motion – based on the extraction of a so-called "effective" pure shearing motion – has been motivated by the fact that vorticity cannot distinguish between pure shearing motions and the actual swirling motion of a vortex. This decomposition technique results in two additive vorticity parts (and, analogously, in two additive strain-rate parts) of distinct nature, namely the *shear* component and the *residual* one. The *residual* vorticity represents a direct measure of the actual swirling motion of a vortex. The new *kinematic* vortex-identification method is discussed on the background of previous methods and general vortex-identification requirements (illustrative examples are included).

Keywords: Decomposition of motion; Flow kinematics; Vortex identification; Vortex-identification criteria; Vortical structures; Vorticity decomposition

1. Introduction

Flow modelling and numerical simulation of turbulent and/or complex vortical flows still lack a generally acceptable definition of a vortex though the understanding of vortex dynamics (i.e. generation, evolution, interaction, and decay of vortical structures) should be based on objective and unambiguous detection schemes. There is no doubt that the physical reasoning for these schemes plays a crucial role.

What is a vortex? Though a vortex intuitively represents a distinct flow phenomenon, the answer to this question is neither simple nor unique, e.g. Lugt (1983). He presents one of the intuitive definitions of a vortex as follows: a vortex is the rotating motion of a multitude of material particles around a common centre. These intuitive definitions often depict a vortex in terms of closed or spiralling streamlines or pathlines, local pressure minima, and isovorticity contours and surfaces. However, spiralling streamlines or pathlines are obtained just for an observer moving with the vortex to be identified, and the existence of a local pressure minimum does not guarantee the existence of a vortex (and vice versa).

Vorticity tensor, as a Galilean invariant quantity (i.e. independent of the translational velocity of an observer) expressing an average angular velocity of fluid elements, appears as one of the most natural choices for a vortex-identification criterial measure. However, it has been recently emphasized by many authors that vorticity is not suitable for the identification of a vortex as it cannot distinguish between pure shearing motions and the actual swirling motion of a vortex (Jeong and Hussain, 1995; Kida and Miura, 1998; Cucitore et al., 1999). This property of

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Nomenclature

$\mathbf{A}$	additive part of $\nabla\mathbf{u}$
C	vortex-strength parameter in (29)
D	jet-nozzle diameter
K	shearing-strength parameter in (28)
L, L_1, L_2	velocity magnitudes, see Fig. 6
p	pressure
P	arbitrary point of the flow field
$\mathbf{Q}$	orthogonal linear transformation
Q	second invariant of $\nabla\mathbf{u}$
r	radius
R	third invariant of $\nabla\mathbf{u}$
s	principal rate of strain in 2D, see (13) and (14)
$\mathbf{S}$, S_{ij}	strain-rate tensor, symmetric part of $\nabla\mathbf{u}$
t	time
$\mathbf{u}$, u_i	velocity vector
u, v, w	components of the velocity vector
U_C	crossflow velocity
$V_{\text{tangential}}$	tangential velocity
x, y, z	coordinates

Greek symbols

α, β, γ	rotation angles
α_1, α_2	characteristic angles in Fig. 2 associated with different vorticity components
β_1, β_2	characteristic angles in Fig. 2 associated with different strain-rate components
Γ	circulation
Δ	vortex-identification criterial quantity defined by (2)
λ	eigenvalue
λ_1, λ_2, λ_3	ordered eigenvalues, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$
ν	kinematic viscosity
ρ	density
σ	shearing parameter in (28)
ω	vorticity-tensor component in 2D, see (13) and (15)
$\mathbf{\Omega}$, Ω_{ij}	vorticity tensor, antisymmetric part of $\nabla\mathbf{u}$

Subscripts and superscripts

BRF	basic reference frame
,	subscript comma denoting differentiation (e.g. $u_{i,j} \equiv \partial u_i/\partial x_j$)
EL	elongation
SH	shearing motion, shear component
SHEARING	shearing motion described by (28)
RES	residual component
RR	rigid-body rotation
VORTEX	vortical motion described by (29)
x, y, z	partial derivatives (e.g. $u_x \equiv \partial u/\partial x$)
T	transpose
*	new Cartesian coordinate (after rotation)

Other symbols and special functions

$\Vert\cdots\Vert$	absolute tensor value, the norm $\Vert\mathbf{G}\Vert$ of any tensor $\mathbf{G}$ is defined by $\Vert\mathbf{G}\Vert = [\mathrm{tr}(\mathbf{G}\mathbf{G}^{\mathrm{T}})]^{1/2}$
∇	nabla operator
MAX (...)	returns the maximum from a given set
MIN (...)	returns the minimum from a given set
sgn ...	returns ±1 according to the sign of a given argument

vorticity leads to the misrepresentation of vortex geometry. Moreover, vortex geometry depends on the vorticity threshold applied.

The most widely used local methods for vortex identification are based on the analysis of the velocity-gradient tensor $\nabla\mathbf{u} = \mathbf{S} + \mathbf{\Omega}$, its symmetric and antisymmetric parts, strain-rate tensor $\mathbf{S}$ and vorticity tensor $\mathbf{\Omega}$, respectively, and the three invariants of $\nabla\mathbf{u}$. Truesdell (1953) was the first to describe a quantitative measure of rotation in terms of $\mathbf{\Omega}$ and $\mathbf{S}$ normalizing the magnitude of the vorticity tensor by the magnitude of the strain-rate tensor (the so-called kinematic vorticity number). However, this measure does not discriminate between vortices with small vorticity in a flow with small shear and vortices with large vorticity in a flow with large shear (Jeong and Hussain, 1995; Geers et al., 2005). The analysis of $\nabla\mathbf{u}$ provides a rational basis for vortex identification and the general classification of 3D flow fields (Chong et al., 1990). The application of complex measures derived from $\nabla\mathbf{u}$ has already revealed its importance for the description of large-scale vortical structures in turbulent free shear flows as well as turbulent boundary-layer flows. The identification of coherent vortices on the basis of direct numerical simulation (DNS) and large-eddy simulation (LES) of turbulent flows has become particularly important (e.g. Dubief and Delcayre, 2000; Lesieur et al., 2003; García-Villalba et al., 2006). In the following section the most popular identification criteria are shortly described. Needless to say, these criteria are Galilean invariant.

In Section 3, some other – more recent – identification methods and discussions about vortex definition are surveyed with a particular emphasis on the applicability limitations of various schemes. New general requirements for vortex identification are summarized and other natural requirements are pointed out.

In Sections 4 and 5, the vortex-identification outcome of the proposed triple decomposition of the relative motion near a point is presented and discussed on the background of previous methods and general identification requirements. Illustrative examples of this novel approach are included.

2. The most widely used local vortex-identification criteria

A vortex obviously represents a non-local flow phenomenon in space and time. However, as noted by Chakraborty et al. (2005), the presence of viscosity in real fluids results in the continuity of the kinematic features of the flow field. Consequently, a reasonable estimate of some non-local vortical features can be inferred from the local (pointwise, applied point by point) methods and characteristics.

2.1. Q-criterion

Hunt et al. (1988) identify vortices of an incompressible flow as connected fluid regions with a positive second invariant of $\nabla\mathbf{u}$ (in tensor notation below the subscript comma denotes differentiation)

$$Q \equiv \frac{1}{2}(u_{i,i}^2 - u_{i,j}u_{j,i}) = -\frac{1}{2}u_{i,j}u_{j,i} = \frac{1}{2}(\|\boldsymbol{\Omega}\|^2 - \|\mathbf{S}\|^2) > 0, \tag{1}$$

that is, as the regions where the vorticity magnitude prevails over the strain-rate magnitude. The norm (or absolute tensor value) $\|\mathbf{G}\|$ of any tensor $\mathbf{G}$ is defined by $\|\mathbf{G}\| = [\mathrm{tr}(\mathbf{G}\mathbf{G}^{\mathrm{T}})]^{1/2}$. In addition, the pressure in the vortex region is required to be lower than the ambient pressure.

2.2. Δ-criterion

Dallmann (1983), Vollmers et al. (1983), and Chong et al. (1990) define vortices as the regions in which the eigenvalues of $\nabla\mathbf{u}$ are complex (a pair of complex-conjugate eigenvalues occurs) and the streamline pattern is spiralling or closed in a local reference frame moving with the point. Such points can be viewed within the critical-point theory – on a plane spanned by the complex eigenvectors – as elliptic ones (focus or centre). For incompressible flows, this requirement reads

$$\Delta = \left(\frac{Q}{3}\right)^3 + \left(\frac{R}{2}\right)^2 > 0, \tag{2}$$

where Q and R are the invariants of $\nabla\mathbf{u}$, Q is given by (1), $R \equiv \mathrm{Det}(u_{i,j})$. Q and R play a key role in the reduced (due to incompressibility) characteristic equation for the eigenvalues λ of $\nabla\mathbf{u}$: $\lambda^3 + Q\lambda - R = 0$.

2.3. λ_2-criterion

The approach of Jeong and Hussain (1995) is formulated on dynamic considerations, namely on the search for a pressure minimum across the vortex. By taking the gradient of the Navier–Stokes equations and by decomposing it into symmetric and antisymmetric parts they derive the well-known vorticity transport equation and the strain-rate transport equation. The latter reads

$$\frac{\mathrm{D}S_{ij}}{\mathrm{D}t} - \nu S_{ij,kk} + \Omega_{ik}\Omega_{kj} + S_{ik}S_{kj} = -\frac{1}{\rho}p_{,ij}, \tag{3}$$

where the pressure Hessian $p_{,ij}$ contains information on local pressure extrema. The occurrence of a local pressure minimum in a plane across the vortex requires two positive eigenvalues of the tensor $p_{,ij}$.

By removing the unsteady irrotational straining and viscous effects from the strain-rate transport equation (3) one yields the vortex-identification criterion for incompressible fluids in terms of two negative eigenvalues of $\mathbf{S}^2 + \boldsymbol{\Omega}^2$. The existence of a local pressure minimum is neither a sufficient nor a necessary condition for the presence of a vortex in general, and the two removed terms from the Eq. (3) are found to be the main cause of this inaccuracy. Finally, a vortex is defined as a connected fluid region with two negative eigenvalues of $\mathbf{S}^2 + \boldsymbol{\Omega}^2$. Since the tensor $\mathbf{S}^2 + \boldsymbol{\Omega}^2$ is symmetric, it has real eigenvalues only. If these eigenvalues are ordered as follows, $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$, the vortex-identification criterion is equivalent to the resulting condition $\lambda_2 < 0$.

3. Other vortex-identification approaches, new requirements and limitations

The papers mentioned in the preceding section have stimulated new and significant research activity during last decade. Some other, more recent, vortex-identification methods and discussions on vortex definition are briefly surveyed below. New requirements for vortex-identification schemes are summarized at the end of this section.

Kida and Miura (1998), similarly to Jeong and Hussain (1995), point out that a swirling motion is not always associated with a sectional pressure minimum. This aspect has motivated these authors to improve the pressure-minimum scheme by imposing a kinematic swirl condition and they have constructed the central axes of vortices by the sectional-swirl-and-pressure-minimum scheme for the identification of the low-pressure vortices in freely decaying homogeneous turbulence. As noted by Wu et al. (2005), an extremal condition adopted by Kida and Miura (1998) for low-pressure vortices is necessary for identifying a single line as the vortex axis.

In the eduction of longitudinal vortices in wall-turbulence, Jeong et al. (1997) employ a non-zero threshold for λ_2 contrary to the λ_2-criterion. This practical aspect is emphasized by Kida and Miura (1998), as well as by Lin et al. (1996) in their study of a neutrally stratified planetary boundary-layer flow.

For compressible flows, the Q-criterion suffers from ambiguity as it offers two ways of extension which have different physical meaning, the second invariant of $\nabla\mathbf{u}$ and the quantity $(\|\boldsymbol{\Omega}\|^2 - \|\mathbf{S}\|^2)/2$. Both choices cannot basically avoid dependence on a non-zero divergence. The λ_2-criterion has been originally tailored for incompressible flows. In the case of compressible fluids, additional terms occur on the LHS of the Eq. (3) as shown by Cucitore et al. (1999) who examined the λ_2-criterion. The use of $\mathbf{S}^2 + \boldsymbol{\Omega}^2$ as an approximation of the pressure Hessian $p_{,ij}$ for compressible fluids requires discarding other terms besides the unsteady irrotational straining and viscous effects originally removed from the strain-rate transport Eq. (3) valid for incompressible flows only. These additional terms are related to a non-zero divergence and non-zero density gradients.

Another aspect worth mentioning is that all the methods based on the analysis of $\nabla\mathbf{u}$ are pointwise providing local vortex-identification criteria. This aspect has led Cucitore et al. (1999) to introduce the concept of non-locality for determining the vortices as structures. They suggest a non-local, Galilean-invariant identification technique. The reason is that the use of local procedures often selects a particular privileged direction which is considered as the vortex axis. For example, the λ_2-criterion captures the pressure minimum in a plane across the vortex but not along it. The non-local criterion of Cucitore et al. (1999) is based on the intuitive notion that the particles inside a vortical structure show small variations in their relative distance even when following completely different trajectories.

Zhou et al. (1999) use the imaginary part of the complex eigenvalue of $\nabla\mathbf{u}$ to visualize vortices and to quantify the strength of the local swirling motion inside the vortex (the so-called swirling-strength criterion). Their method is based on the Δ-criterion, however, it identifies not only the vortex region (equivalently as the Δ-criterion), but also the local strength and the local plane of swirling. For a similar approach based on complex eigenvalues, see Berdahl and Thompson (1993). The associated instantaneous-streamline analysis is further developed in Chakraborty et al. (2005). Their criterion enhances the swirling-strength criterion by including a local approximation of the non-local property proposed by Cucitore et al. (1999), requiring that the swirling material points inside a vortex have bounded separation remaining small. Moreover, Chakraborty et al. (2005) study the relationship between local identification schemes. They show that all of the discussed local criteria, given the proposed usage of threshold, result in a remarkable vortex similarity.

The evaluation of vortex-identification criteria by comparing the resulting vortex patterns in numerically simulated complex vortical flows cannot lead to a final choice for the correct criterion as the judgment depends on the adopted intuitive and subjective concept of the investigator on what should be called a vortex, as pointed out by Wu et al. (2005). Instead of numerical examples they make an analytical diagnosis of four local criteria, demonstrated by the Burgers and Sullivan vortex, indicating that Q-criterion and λ_2-criterion may cut a connected vortex into broken segments at locations with strong axial stretching. Consequently, the following requirements are emphasized: a generally applicable vortex definition should be able to identify the vortex axis and allow for an arbitrary axial strain. Note that all of the local criteria described in Section 2 identify just a vortex region through criterial inequalities without specifying the vortex axis inside this region. Wu et al. (2005) state that an equality is necessary for identifying a single line as the vortex axis. Recall that the extremal condition of Kida and Miura (1998) enables us to find the vortex skeleton by tracing the lines of the sectional pressure minimum, provided that a swirl condition is satisfied.

Xiong et al. (2004) define a vortex as follows: if all the fluid particles within the area in the plane normal to the vorticity direction have the rotational velocity components in the same direction around any point in the area, the area is identified as a part of a vortex. They clearly state that the validation of the vortex definition using flow examples is not rigorous because of a prejudice of expecting what is *a priori* a vortex. Examples only suffice to invalidate but not support an idea. Xiong et al. (2004) emphasize that a method of vortex identification should be evaluated by whether there is a clear physical meaning in the vortex definition and that the vortex-identification method should be chosen by which characteristics the definition focuses on.

The fact that all of the most widely used vortex definitions are not *objective* relative to an arbitrarily rotating reference frame has motivated Haller (2005) to develop an *objective* frame-independent definition of a vortex based on Lagrangian stability considerations. An incompressible vortex is defined as a set of fluid trajectories along which the strain acceleration tensor is indefinite over directions of zero strain. This definition should help in situations where there is an unclear choice for a reference frame (for example, vortical flows in rotating tanks).

Zhang and Choudhury (2006) formulate a Galilean-invariant scheme, eigen helicity density, based on the concept of Galilean-variant helicity density adopted earlier by Levy et al. (1990) for the graphical representation of 3D flow fields that contain concentrated vortices. Their scheme shows a promise to identify vortices in 3D compressible, variable-density flows governed by the baroclinic term (i.e. the normalized cross product of a density gradient and pressure gradient) in the vorticity equation.

The above mentioned new requirements for vortex-identification schemes and their underlying criterial quantities can be summarized as follows:

- validity for compressible flows,
- validity for variable-density flows,
- avoidance of the subjective choice of threshold in the vortex-boundary identification,
- determination of the local intensity of the swirling motion (to describe the inner structure of a vortex),
- vortex-axis identification,
- allowance for an arbitrary axial strain,
- non-local properties,
- ability to provide the same results in different rotating frames (to fulfil *material objectivity* or *frame indifference*, i.e. both translational and rotational independence, see e.g. Leigh, 1968).

Though most of these requirements are intuitively clear, some of them may need further justification. For example, the allowance for an arbitrary axial strain has become a subject of recent debate (Chakraborty et al., 2006; Wu et al., 2006).

As to the local intensity of the swirling motion, engineering practice may frequently need this local quantity to be easily integrable across the vortex region in order to obtain the integral strength of a vortex. However, the application of conventional circulation Γ (calculated as a surface quadrature of vorticity) for this purpose is in fact misleading as the vorticity is misrepresenting the local intensity of the actual swirling motion of a vortex. For example, one obtains a net circulation for the region of a simple linear shearing motion due to a net vorticity.

Furthermore, another requirement – much more trivially looking than the integral strength of a vortex – is swirl orientation. Vorticity may answer the question regarding the swirl orientation in simple problems, however, the local angle between the vortex-axis tangent and the vorticity vector may reach large values due to a strong shearing aligned with the vortex axis (e.g. streamwise vortical structures in a turbulent boundary layer). This requirement becomes particularly important in complex 3D vortical flows subjected to high shear.

As noted by Kida and Miura (1998), it is impossible for the isosurface representation of a scalar field (applied in vortex identification) to distinguish between individual vortical structures. This is especially the problem of homogeneous turbulence. To avoid ambiguity and vortex overlaps, the explicit vortex-axis requirements are proposed below.

These additional requirements are quite natural and, therefore, added to those already mentioned:

- swirl orientation,
- determination of the (integral) vortex strength,
- vortex-axis requirements: existence and uniqueness for each connected vortex region (to avoid ambiguity and vortex overlaps).

4. Triple decomposition of the relative motion near a point and vortex identification

4.1. Triple decomposition of the relative motion near a point

The conventional double decomposition of motion near a point into a pure irrotational straining motion along the principal axes of the rate of strain tensor (generally including a uniform dilatation) and a rigid-body rotation expressed by $\nabla\mathbf{u} = \mathbf{S} + \mathbf{\Omega}$ has a long history and stems from the Cauchy–Stokes decomposition theorem (first explicitly stated by Stokes, 1845, according to Truesdell and Toupin, 1960). Basic kinematics in this regard can be found in many textbooks (e.g. Batchelor, 1967; Panton, 1984).

The triple decomposition of the relative motion near a point (TDM) has been motivated by the fact that vorticity cannot distinguish between pure shearing motions and the actual swirling motion of a vortex. Analogously, strain rate cannot distinguish between straining motions and shearing motions. These problems indicate that the double decomposition may not satisfy all of today's needs of fluid mechanics. The aim of the TDM is to decompose an arbitrary instantaneous state of the relative motion near a point into three elementary motions, each described by an additive part of $\nabla\mathbf{u}$ with a distinct tensor character, explicitly including a pure shearing motion. Therefore, the present decomposition method is – including its vortex-identification outcome – based on the extraction of a so-called "effective" pure shearing motion. The TDM is expressed through the corresponding triple decomposition of $\nabla\mathbf{u}$ introduced in Kolář (2004).

To discuss all of the main aspects of the TDM in 3D flows is far beyond the scope of this contribution and the research in this regard is incomplete due to its complexity (see also the final remark at the end of this subsection). However, its planar version – including the application to vortex identification – is very illustrative. Particularly, a straightforward comparison with the most widely used vortex-identification methods is provided for planar flows in Section 5.

The qualitative model of three elementary motions of the TDM is depicted in Fig. 1. The deformable fluid element in Fig. 1 consists of discrete undeformable material points in terms of which the local rate of deformation is described through their relative motion. The material point represents – in the present context – "much less than a fluid element" and generally allows translation and rotation only. A pure shearing motion near a point is interpreted in terms of the parallel relative motion of non-rotating undeformed shearing elements – planes, lines, or points (depending on flow complexity in 3D).

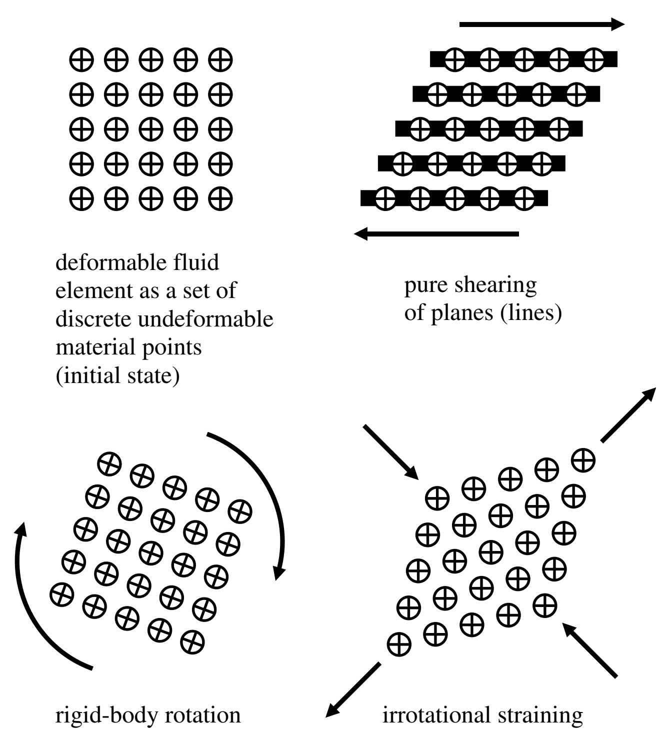

Fig. 1. Qualitative model of three elementary motions of the TDM.

A pure shearing motion in Fig. 1 is not a mere combination of an irrotational straining motion with a rigid-body rotation as in the case of the double decomposition. This fact can be easily checked through the rotational change of material points which remains zero for a pure shearing motion within the proposed qualitative model of the TDM. The rotational change of material points is just the quantity reflecting the actual swirling motion of a vortex: note that both an irrotational straining and a pure shearing motion do not contribute to this rotational change (at least according to the present approach).

Although we focus below on planar flows, it is convenient to introduce the quantitative TDM algorithm using 3D formalism. The TDM distinguishes three different elementary motions near a point, each defined in terms of a distinct tensor structure. The TDM reads

$$\nabla\mathbf{u} = (\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}} + (\nabla\mathbf{u})_{\mathrm{SH}}, \tag{4}$$

where an irrotational straining motion is given by the symmetric tensor $(\nabla\mathbf{u})_{\mathrm{EL}}$ (subscript "EL" denotes elongation), a rigid-body rotation (denoted by "RR") is given by the antisymmetric tensor $(\nabla\mathbf{u})_{\mathrm{RR}}$, and an *effective* pure shearing motion (denoted by "SH") is described by the "purely asymmetric tensor form" $(\nabla\mathbf{u})_{\mathrm{SH}}$ its components $u_{i,j}$ fulfilling *in a suitable reference frame*

$$u_{i,j} = 0 \quad \text{OR} \quad u_{j,i} = 0 \quad (\text{for all } i, j) \tag{5}$$

with the implication

$u_{i,j}u_{j,i} = 0$ (no summation).

The condition (5) requires zeros on the leading diagonal and, moreover, at least one off-diagonal element from each pair must be zero as well. If the frame showing the tensor in form (5) exists, the tensor is – by definition – *purely asymmetric*. The tensor structures represented by (5) are, for example,

$$\begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ \bullet & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & \bullet \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & 0 \\ 0 & \bullet & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & \bullet \\ 0 & 0 & 0 \\ 0 & \bullet & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & 0 \\ 0 & 0 & \bullet \\ \bullet & 0 & 0 \end{pmatrix}.$$

While the symmetric-tensor condition (specifying an irrotational straining motion)

$$u_{i,j} = u_{j,i} \quad (\text{for all } i, j) \tag{6}$$

with the implication

$u_{i,j}u_{j,i} \geqslant 0$ (no summation)

and the antisymmetric-tensor condition (specifying a rigid-body rotation)

$$u_{i,j} = -u_{j,i} \quad (\text{for all } i, j) \tag{7}$$

with the implication

$u_{i,j}u_{j,i} \leqslant 0$ (no summation)

are fulfilled *in an arbitrary reference frame* rotated under an orthogonal transformation, the condition (5) for the *purely asymmetric* tensor is satisfied *in a suitable reference frame only*. In this reference frame, a straightforward consequence of (5) is that the strain-rate and vorticity magnitudes are in strict equilibrium, component by component, as the relation $|S_{ij}| = |\Omega_{ij}|$ holds for all i, j. Further, the condition (5) implies the equality $\|\mathbf{S}\| = \|\boldsymbol{\Omega}\|$ valid in an arbitrary reference frame (it does not hold vice versa). The condition (5) defines – at least within the present paper – the *purely asymmetric* tensor with respect to second-order tensors and, moreover, it defines a general structure of a pure shearing motion with respect to flow kinematics near a point.

The above stated decomposition requirements are apparently insufficient. Considering $(\nabla\mathbf{u})_{\mathrm{SH}}$, the condition (5) is a necessary condition only as it characterizes just a pure shearing motion without specifying the label "*effective*". What we really need is a physically well-justified algorithm leading to a unique decomposition. The "shear tensor" satisfying the condition (5) can be easily generated by the natural and straightforward decomposition scheme which is applicable to an arbitrary reference frame

$$\nabla\mathbf{u} \equiv \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = \begin{pmatrix} \text{residual} \\ \text{tensor} \end{pmatrix} + \begin{pmatrix} \text{shear} \\ \text{tensor} \end{pmatrix}, \tag{8a}$$

where the residual tensor is given by

$$\begin{pmatrix} \text{residual} \\ \text{tensor} \end{pmatrix} = \begin{pmatrix} u_x & (\operatorname{sgn} u_y)\mathrm{MIN}(|u_y|,|v_x|) & \bullet \\ (\operatorname{sgn} v_x)\mathrm{MIN}(|u_y|,|v_x|) & v_y & \bullet \\ \bullet & \bullet & w_z \end{pmatrix}. \tag{8b}$$

In (8a,b) the following simplified notation is employed: u, v, w are velocity components, subscripts x, y, z stand for partial derivatives. The remaining two non-specified pairs of off-diagonal elements of the residual tensor in (8b) are constructed strictly analogously as the specified one, each pair being either symmetric or antisymmetric. Just a simple quantitative example:

$$\begin{pmatrix} -1 & 15 & 17 \\ 3 & 8 & -14 \\ -26 & -14 & -5 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 17 \\ 3 & 8 & -14 \\ -17 & -14 & -5 \end{pmatrix} + \begin{pmatrix} 0 & 12 & 0 \\ 0 & 0 & 0 \\ -9 & 0 & 0 \end{pmatrix}.$$

If considered separately, an arbitrary shearing motion should be, by (5), recognized *in a suitable reference frame* as a third elementary part of the TDM. Finding a minimum of the norm of the residual tensor within the scheme (8a,b) by changing the reference frame under an orthogonal transformation guarantees to satisfy this necessary requirement and leads to the correct frame choice (to perform the desired decomposition). This condition says that the effect of the extraction of the shear tensor from $\nabla\mathbf{u}$ is maximized.

The TDM is closely associated with the so-called *basic reference frame* (BRF) where it is performed as the three separate parts of $\nabla\mathbf{u}$ are generated just here through the decomposition scheme (8a,b), where the residual tensor is to be further decomposed into its symmetric and antisymmetric parts representing $(\nabla\mathbf{u})_{\mathrm{EL}}$ and $(\nabla\mathbf{u})_{\mathrm{RR}}$. However, the decomposition results generated in the BRF are valid for all other frames rotated (not rotating!) with respect to the BRF under an orthogonal transformation. In the BRF, an *effective* pure shearing motion is shown in a clearly visible manner described by the form (5) on condition that the norm of the residual tensor in (8a,b) is minimized. That is, consistently with the limiting case mentioned in the preceding paragraph, on condition that the effect of the extraction of the shear tensor is maximized. This is the reason to label the obtained shearing motion with the term "*effective*". Considering the following relation expressed in terms of the strain-rate and vorticity tensors, $\mathbf{S}$ and $\boldsymbol{\Omega}$, and valid in an arbitrary reference frame,

$$\left\|\begin{pmatrix}\text{residual}\\ \text{tensor}\end{pmatrix}\right\|^2 + 4(|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|) = \|\nabla\mathbf{u}\|^2, \tag{9}$$

where $\|\nabla\mathbf{u}\|$ remains unchanged under an orthogonal transformation and, consequently, the definition condition of the BRF takes the form

$$[|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|]^{\mathrm{BRF}} = \mathrm{MAX}([|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|]^{\text{ALL FRAMES}}). \tag{10}$$

The condition (10) says that – by changing the reference frame under an orthogonal transformation $\mathbf{Q}$ – we are choosing from all frames mutually rotated (not rotating!) the frame in which the quantity $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$ attains its maximum. In practice, though the number of frames is infinite, the determination of the BRF should be based on a finite set of discrete frame representations. By considering a reasonable angle resolution (for example, one degree or less) we proceed to the BRF approximation with reasonably high precision. The transformation matrix $\mathbf{Q}$ for an arbitrarily rotated Cartesian coordinate system can be obtained by a sequence of three rotational transformations, see Appendix A.

The orientation of the BRF (and, consequently, the orientation of an *effective* pure shearing motion) is a local aspect of the flow field, similarly as the orientation of principal axes. However, note that the BRF, unlike the system of principal axes of $\mathbf{S}$, is determined *simultaneously* on the basis of $\mathbf{S}$ and $\boldsymbol{\Omega}$ through the "interaction scalar" $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$.

It is worth mentioning that the application of a qualitatively different frame-choice criterion maximizing directly the magnitude of a pure shearing motion, that is, maximizing $\left\|\begin{pmatrix}\text{shear}\\ \text{tensor}\end{pmatrix}\right\|$, leads to an ambiguous decomposition algorithm, even for 2D fluid motion. This criterion takes into account only the magnitude of a pure shearing motion, not its effect (i.e. its impact on $\nabla\mathbf{u}$ after the extraction) in full which includes the shearing structure and orientation of shearing elements (planes, lines, or points). Further, by changing the present frame-choice criterion based on $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$ quantitatively to the other extreme value, namely by choosing the shear-free frame by requiring $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}| = 0$, the decomposition scheme (8a,b) leads to the well-known double-decomposition results. In this case, the principal axes of $\mathbf{S}$ represent the desired shear-free frame (though this frame is not always the only shear-free frame for a given tensor data).

The TDM algorithm consists of the following three steps (a uniform dilatation does not affect the interaction scalar in the condition (10) and can be removed prior to a further analysis of $\nabla\mathbf{u}$ without loss of generality and applicability to compressible flows):

Step 1: Determination of the BRF satisfying the condition (10).
Step 2: Decomposition of $\nabla\mathbf{u}$ following the scheme (8a,b); according to the initial scheme (4), the residual tensor represents the sum $(\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}}$, and the shear tensor represents $(\nabla\mathbf{u})_{\mathrm{SH}}$.
Step 3: Return to the original (e.g. laboratory) reference frame: any additive part $\mathbf{A}_i$ of $\nabla\mathbf{u}$ is described in an arbitrary reference frame rotated (not rotating!) with respect to the BRF under an orthogonal transformation $\mathbf{Q}$ by $\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}$ as

$$\mathbf{Q}(\nabla\mathbf{u})\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\left(\sum_i \mathbf{A}_i\right)\mathbf{Q}^{\mathrm{T}} = \sum_i \mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}. \tag{11}$$

As mentioned earlier, the BRF is a local frame and its orientation generally changes from point to point (at a given instant in time). If we wish to see the whole field of relevant quantities in a common reference frame (e.g. laboratory reference frame) we have to do Step 3.

In the planar case treated below in detail, the uniqueness of the TDM is obvious.

From the viewpoint of the double decomposition, the TDM components of $\nabla\mathbf{u}$ are certain products of interaction between $\mathbf{S}$ and $\boldsymbol{\Omega}$. Unlike the two elementary parts of the double decomposition, the three elementary parts of the TDM are mutually conditionally balanced. The term $(\nabla\mathbf{u})_{\mathrm{SH}}$ is responsible for a specific portion of vorticity labelled "*shear* vorticity" and for a specific portion of strain rate labelled "*shear* strain rate". The remaining portions are called "*residual* vorticity" and "*residual* strain rate". For the quantitative relation between the TDM and the double decomposition it holds (subscripts "SH" and "RES" by $\mathbf{S}$ and $\boldsymbol{\Omega}$ denote their *shear* and *residual* components, respectively)

$$
\begin{aligned}
\nabla\mathbf{u} &= (\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}} + (\nabla\mathbf{u})_{\mathrm{SH}} \\
&= (\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}} + \frac{1}{2}[(\nabla\mathbf{u})_{\mathrm{SH}} + ((\nabla\mathbf{u})_{\mathrm{SH}})^{\mathrm{T}}] \\
&\quad + \frac{1}{2}[(\nabla\mathbf{u})_{\mathrm{SH}} - ((\nabla\mathbf{u})_{\mathrm{SH}})^{\mathrm{T}}] \\
&= \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + \mathbf{S}_{\mathrm{SH}} + \mathbf{\Omega}_{\mathrm{SH}} \\
&= [\mathbf{S}_{\mathrm{RES}} + \mathbf{S}_{\mathrm{SH}}] + [\mathbf{\Omega}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{SH}}] = \mathbf{S} + \mathbf{\Omega}. \qquad (12)
\end{aligned}
$$

In the triple decomposition of $\nabla\mathbf{u}$, $\mathbf{S}$ and $\mathbf{\Omega}$ are cut down in magnitudes to "share" their portions through the third term $(\nabla\mathbf{u})_{\mathrm{SH}}$ associated with a pure shearing motion as $\nabla\mathbf{u} = \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + (\nabla\mathbf{u})_{\mathrm{SH}}$. The proposed tensor decomposition is applicable to general second-order Cartesian tensors and may prove its usefulness to problems outside fluid mechanics.

In planar incompressible flows, the velocity-gradient tensor $\nabla\mathbf{u}$ can be described in an arbitrary reference frame, in the system of principal axes, and in the above mentioned BRF as follows:

$$
\begin{pmatrix} u_x & u_y & 0 \\ v_x & -u_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} s & -\omega & 0 \\ \omega & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\text{PRINCIPAL AXES}} \rightarrow \begin{pmatrix} 0 & s-\omega & 0 \\ s+\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\text{BRF}}, \qquad (13)
$$

where s (i.e. the 2D principal rate of strain) and ω (i.e. the vorticity-tensor component in 2D) fulfil the relations

$$
|s| = \left(\sqrt{4u_x^2 + (u_y + v_x)^2}\right)\Big/2, \qquad (14)
$$

$$
\omega = (v_x - u_y)/2. \qquad (15)
$$

In the BRF, there are two different relative rotational orientations of $u_y\,\mathrm{d}y$ and $v_x\,\mathrm{d}x$, the same and the opposite, see Fig. 2. In Fig. 2a vorticity dominates strain rate. The characteristic angles, α_1 and α_2, correspond to the *residual* vorticity ω_{RES} (associated with the rigid-body rotation) and *shear* vorticity ω_{SH} (associated with the pure shearing motion) while their sum is proportional to the *total* vorticity ω. In Fig. 2b strain rate dominates vorticity. The characteristic angles, β_1 and β_2, correspond to the *residual* strain rate s_{RES} and *shear* strain rate s_{SH} while their sum is proportional to the *total* strain rate s. Note that ω_{RES} and ω_{SH} have the same signs in the BRF due to the algebraic structure of (8a,b). The same holds for s_{RES} and s_{SH}. The same signs indicate a *non-destructive* nature of the superimposing construction in Fig. 2. The virtual superposition is applicable to infinitesimal motional changes only. For both rotational orientations, the magnitude of the superimposed shearing motion is given by the difference of the absolute values of u_y and v_x. In planar flows, a non-zero ω_{RES} apparently existing only for the same rotational orientation of $u_y\,\mathrm{d}y$ and $v_x\,\mathrm{d}x$, see Fig. 2, excludes the existence of a non-zero s_{RES} existing only for the opposite rotational orientation of $u_y\,\mathrm{d}y$ and $v_x\,\mathrm{d}x$.

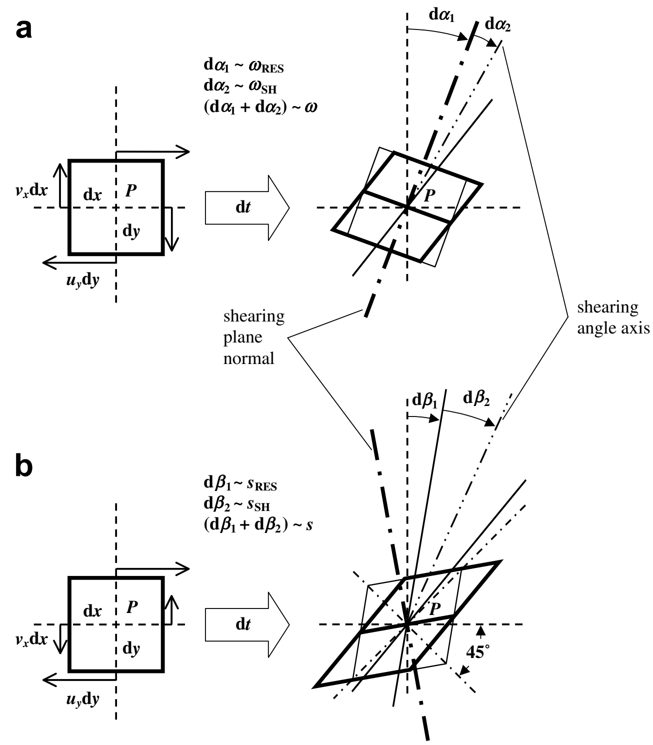

Fig. 2. Geometrical interpretation of 2D fluid motion (incompressible flow) and the TDM outcome: (a) vorticity components, (b) strain-rate components.

By combining the qualitative model of elementary motions of the TDM depicted in Fig. 1 with the situation in 2D flows according to Fig. 2a, it follows that the *residual* vorticity can be interpreted in terms of (twice) the angular velocity of material points due to the rigid-body rotation, the *shear* vorticity in terms of (twice) the average angular velocity of the fluid element due to the shearing part of motion only while the conventional vorticity in terms of (twice) the average angular velocity of the fluid element.

Elementary parts of the TDM and flow patterns for various flow situations in 2D fluid motion (incompressible flow) are shown in Fig. 3. All possible flow configurations near a point for fixed u_y and variable v_x are depicted in the corresponding BRFs showing an *effective* pure shearing motion in a clearly visible manner. The reference points themselves can be described as critical points and the local flow patterns correspond to the leading terms of a Taylor series expansion for the velocity field in terms of space coordinates (Perry and Chong, 1987; Chong et al., 1990).

In 2D flows, with respect to (13) showing explicitly the desired tensor components in the BRF in terms of s and ω, with respect to the algebraic structure of the decomposition (8a,b), and assuming $|s| \geqslant |\omega|$ or $|s| \leqslant |\omega|$ the following set of relations can be derived for s and ω, and their *residual* and *shear* components (the case $|s| = |\omega|$ represents a simple shear)

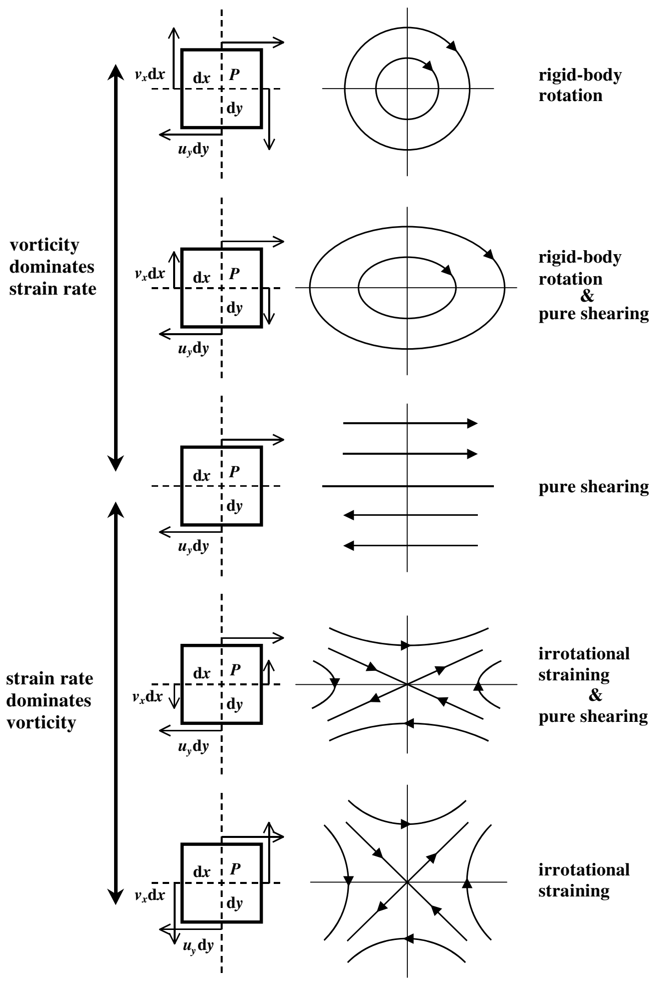

Fig. 3. Elementary parts of the TDM and flow patterns for various flow situations in 2D fluid motion (incompressible flow).

$$s = s_{\mathrm{RES}} + s_{\mathrm{SH}}, \tag{16}$$
$$|s| = |s_{\mathrm{RES}}| + |s_{\mathrm{SH}}|, \tag{17}$$
$$s_{\mathrm{SH}} = (\operatorname{sgn} s)|\omega| \quad \text{for } |s| \geqslant |\omega|, \tag{18}$$
$$s_{\mathrm{SH}} = s \quad \text{for } |s| \leqslant |\omega|, \tag{19}$$
$$s_{\mathrm{RES}} = s - s_{\mathrm{SH}} = (\operatorname{sgn} s)[|s| - |\omega|] \quad \text{for } |s| \geqslant |\omega|, \tag{20}$$
$$s_{\mathrm{RES}} = s - s_{\mathrm{SH}} = 0 \quad \text{for } |s| \leqslant |\omega|, \tag{21}$$
$$\omega = \omega_{\mathrm{RES}} + \omega_{\mathrm{SH}}, \tag{22}$$
$$|\omega| = |\omega_{\mathrm{RES}}| + |\omega_{\mathrm{SH}}|, \tag{23}$$
$$\omega_{\mathrm{SH}} = \omega \quad \text{for } |s| \geqslant |\omega|, \tag{24}$$
$$\omega_{\mathrm{SH}} = (\operatorname{sgn} \omega)|s| \quad \text{for } |s| \leqslant |\omega|, \tag{25}$$
$$\omega_{\mathrm{RES}} = \omega - \omega_{\mathrm{SH}} = 0 \quad \text{for } |s| \geqslant |\omega|, \tag{26}$$
$$\omega_{\mathrm{RES}} = \omega - \omega_{\mathrm{SH}} = (\operatorname{sgn} \omega)[|\omega| - |s|] \quad \text{for } |s| \leqslant |\omega|. \tag{27}$$

As can be inferred from (27), the magnitude of planar *residual* vorticity represents nothing but the simplest measure of the dominance of planar vorticity magnitude over planar strain-rate magnitude. Analogously, from (20), the magnitude of planar *residual* strain rate is nothing but the simplest measure of the dominance of planar strain-rate magnitude over planar vorticity magnitude.

The final remark in this subsection deals with general 3D data. Extensive numerical tests of 3D tensor data indicate that the BRF is unique for local 3D flows near a point without planar or rotational symmetries of $\nabla\mathbf{u}$ at the point. However, even assuming tensor data leading to different BRFs, the TDM may still be unique provided that different BRFs lead to the same TDM results. For example, an arbitrary symmetric or antisymmetric tensor is associated with an infinite number of BRFs, each showing the same decomposition results. The 3D aspects of the TDM need further investigation.

4.2. Vortex identification

The vortex-identification outcome of the TDM reads: a vortex is defined as a connected fluid region with a non-zero magnitude of the *residual* vorticity. A vortex axis is given in a plane across the vortex (but not along it) by the point of a maximum magnitude of the *residual* vorticity vector which is a tangent of the vortex axis at this point. In the planar-flow context, using (26) and (27), the vortex regions are characterized by the non-zero *residual* vorticity for $|s| < |\omega|$, see the application (Kolář, 2004) to the plane turbulent near wake of two side-by-side square cylinders (data from Kolář et al., 1997). It should be mentioned that the *residual* vorticity in 2D problems does not change its orientation perpendicular to the flow plane. Therefore Steps 1 and 3 of the TDM algorithm do not need to be performed. However, considering the strain-rate decomposition even for a planar case, if we are interested not only in the magnitude of strain-rate components as given by (18)–(21) but also in the orientation of shearing planes, we cannot avoid Steps 1 and 3 of the TDM algorithm.

Fig. 4 shows the quasi-2D secondary-flow vortex geometry, namely the contrarotating vortex pair, in three downstream cross-sections of a single jet in crossflow (JICF), data from Kolář et al. (2000). A low threshold applied in Fig. 4 has no physical but numerical reasons only (to avoid staircase outer contours of the *residual* vorticity). Also note that for an arbitrarily chosen threshold level the region of the *residual* vorticity forms a subdomain of the vorticity region. The JICF vortex geometry is on average nearly circular in terms of (streamwise) ω_{RES} somewhat resembling the geometry of an ideal isolated single-fluid viscous vortex. It can be inferred from Fig. 4 that a prolonged shape of (streamwise) vorticity contours is closely associated with the relatively strong shearing effect of the flow through the "gap" between vortices along the centreline of the contrarotating vortex pair. Unlike the *residual* vorticity, the total vorticity of the inner vortex region (approximately between the *residual*-vorticity maximum and the centreline of the contrarotating vortex pair) inevitably absorbs this shearing effect by increasing its magnitude. Consequently, the transverse (i.e. y) vorticity gradient of the inner vortex region is higher in comparison with the outer vortex region which is relatively less affected by the "gap-like flow".

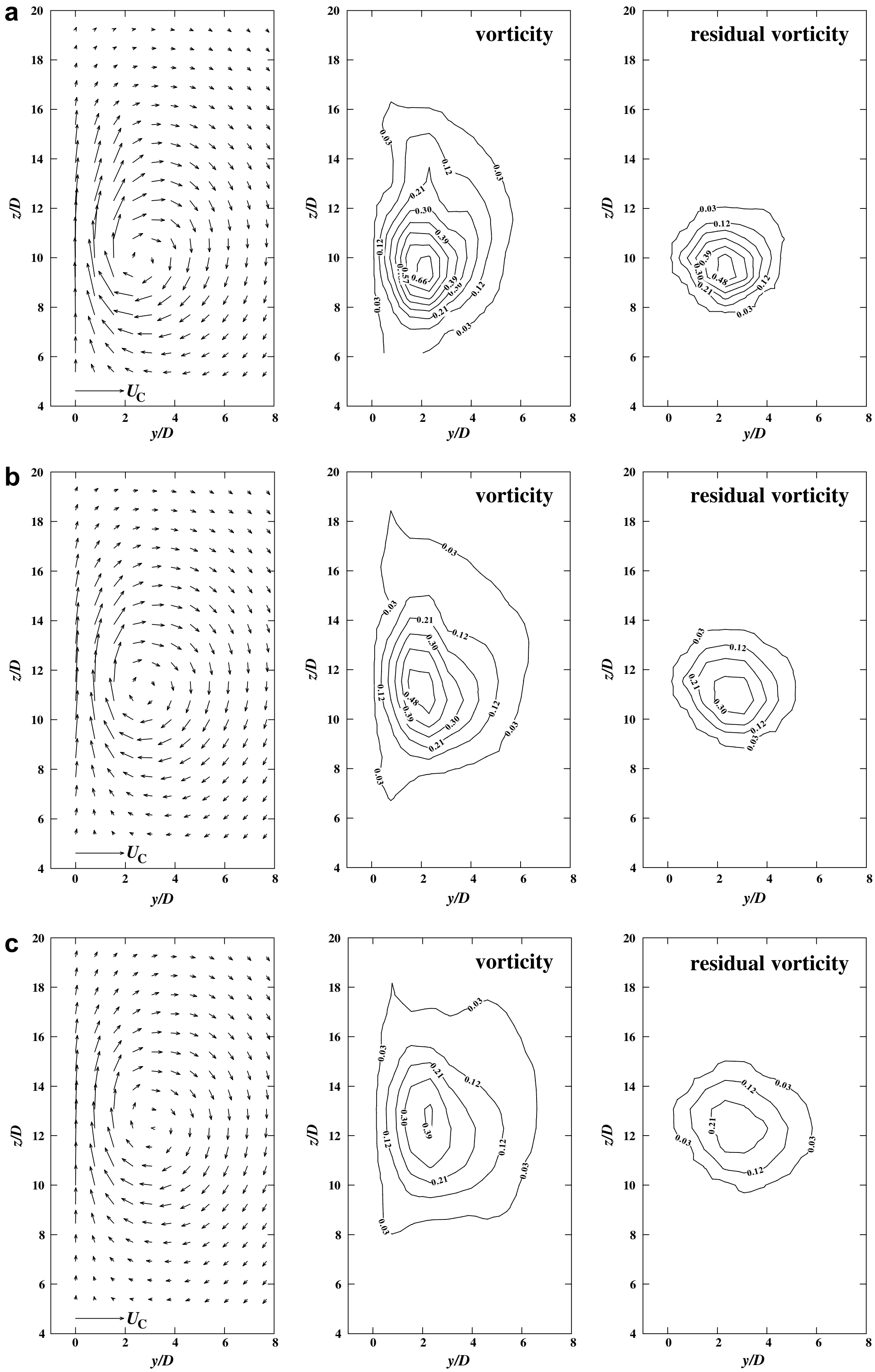

Fig. 4. Secondary-flow vortex geometry of the JICF in terms of velocity, (negative) vorticity and *residual* vorticity in the cross-section at: (a) $x/D = 10$, (b) $x/D = 15$, (c) $x/D = 20$. U_C denotes crossflow velocity, x is crossflow direction (the origin of coordinates is located at the center of the jet exit), D is jet-nozzle diameter.

The employed additive vorticity decomposition implies a corresponding surface-quadrature decomposition. The circulation (strictly said, a portion of total circulation Γ) based on *residual* vorticity can be obtained by integration. This quantity represents the (integral) vortex strength.

5. Discussion

The idea of vorticity decomposition has its own history, though much shorter than the decomposition of motion. It is almost 50 years old, see Astarita (1979), Wedgewood (1999) and the references therein. The result of the Giesekus–Harnoy–Drouot decomposition (this terminology is adopted following Wedgewood, 1999), namely the *objective* portion of $\mathbf{\Omega}$ obtained with respect to the principal axes of $\mathbf{S}$, is proposed by Astarita (1979) for a flow classification. Wedgewood (1999) derives a new vorticity decomposition into two parts, the so-called deformational vorticity and the rigid vorticity. His analysis employs the cross product of a particle's velocity and acceleration, $\mathbf{u} \times \mathrm{D}\mathbf{u}/\mathrm{D}t$, and leads to the evolution equation for the *objective* deformational vorticity. The solution of the 'Wedgewood equation' which depends on both space and time derivatives of $\nabla\mathbf{u}$ is proposed to develop *objective* constitutive equations for complex rheological fluids.

It should be emphasized that the present vorticity decomposition into the *shear* vorticity and the *residual* one is based on the decomposition of motion through the decomposition of a $\nabla\mathbf{u}$-field "frozen" at a given instant in time. The corresponding vortex-identification method represents a certain qualitative "comeback" of vorticity, namely its specific portion, the *residual* vorticity. In view of the requirements summarized in Section 3, note that the *residual* vorticity retains all of the very useful vorticity features – Galilean invariance, vectorial character (direction and orientation), applicability to compressible flows and variable-density flows, easy integrability across an arbitrary surface area, etc. – and satisfies most of the general requirements for vortex identification. Unlike vorticity, the *residual* vorticity distinguishes between pure shearing motions and the actual swirling motion of a vortex and, consequently, it correctly captures vortical motion near a wall by diminishing to zero at the wall. On the other hand, the *residual* vorticity is still a local quantity, moreover, not invariant with respect to rotating reference frames.

In planar incompressible flows, all of the local criteria described in Section 2 degenerate to the same one (Jeong and Hussain, 1995; Wu et al., 2005) identifying the vortex region by the condition $\omega^2 - s^2 > 0$ implying that vorticity dominates strain rate. This criterion corresponds to the Weiss criterion (Weiss, 1991; Basdevant and Philipovitch, 1994) for elliptic flow regions and can be geometrically interpreted as the region of positive unnormalized Gaussian curvature of the stream function (Dresselhaus and Tabor, 1989). The swirling strength of Zhou et al. (1999) is equal in 2D motion to $\sqrt{\omega^2 - s^2}$. The conventional identifier, positive $\omega^2 - s^2$, and the proposed *residual* vorticity are examined below as possible candidates for vortex-axis identification and for characterizing the local intensity of a vortex. Due to the two-dimensionality of the testcase examined below, the term "vortex centre" is employed instead of "vortex axis".

The kinematically consistent model of a planar vortex-shear interaction employed below provides a fair basis for obtaining correct qualitative results in vortex identification. The examined planar $\nabla\mathbf{u}$-field is formed by superimposing two linear shearing effects of a Gaussian distribution located symmetrically at $y = \pm 0.5$ given by

$$u_y^{\mathrm{SHEARING}} = -K\left(\exp\left[-\left(\frac{y-0.5}{\sigma}\right)^2\right] + \exp\left[-\left(\frac{y+0.5}{\sigma}\right)^2\right]\right), \tag{28}$$

onto the $\nabla\mathbf{u}$-field of an ideal axisymmetric (counterclockwise) Taylor vortex centred at (0,0) described by the tangential velocity distribution of the form ($r = \sqrt{x^2 + y^2}$)

$$V_{\mathrm{tangential}}^{\mathrm{VORTEX}} = Cr\exp(-r^2). \tag{29}$$

All parameters in (28) and (29) are positive. The vortex-strength parameter C was set for simplicity to be $C = 1$ and the shearing parameter σ was set to be $\sigma = 1/4$ (both parameters are fixed within the present analysis). The placing of two shearing effects makes the examination procedure much more illustrative than using only one.

Three vortex characteristics – vorticity, positive $\omega^2 - s^2$ (i.e. the conventional planar-flow vortex-identification measure) and the *residual* vorticity – are compared in Fig. 5 for different values of the shearing-strength parameter K, namely $K = 0$, 2, 10. The distribution of positive $\omega^2 - s^2$ exhibits the formation of a double peak quite similar to that of vorticity distribution. For large shearing values, $K \gg 1$, the peak magnitudes of both vorticity and positive $\omega^2 - s^2$ are adequately large and the locations of peak values ultimately attain the locations of shearing maxima at $y = \pm 0.5$. This causes an inevitable ambiguity in defining the vortex centre in terms of vorticity as well as $\omega^2 - s^2$ by using the natural extremal condition, as discussed in Section 3. The identification of a vortex in terms of the non-zero *residual* vorticity indicates that this criterial quantity provides an identical vortex boundary as the positive $\omega^2 - s^2$. However, there is no ambiguity in defining the vortex-centre location as the characteristic peak value of the *residual* vorticity remains at the original centred position (for $K = 0$) independently of the strength of a superimposed shearing and its magnitude remains for $K \gg 1$ unchanged as well.

It is shown in Fig. 5 that the conventional criterial quantity, positive $\omega^2 - s^2$, satisfactorily identifying the overall vortex region fails – unlike ω_{RES} – to describe the vortex centre correctly (in terms of the location and magnitude of the peak value) provided that a pure shearing motion superimposed onto an ideal axisymmetric vortex is not negligible. This quantity cannot represent the local intensity of

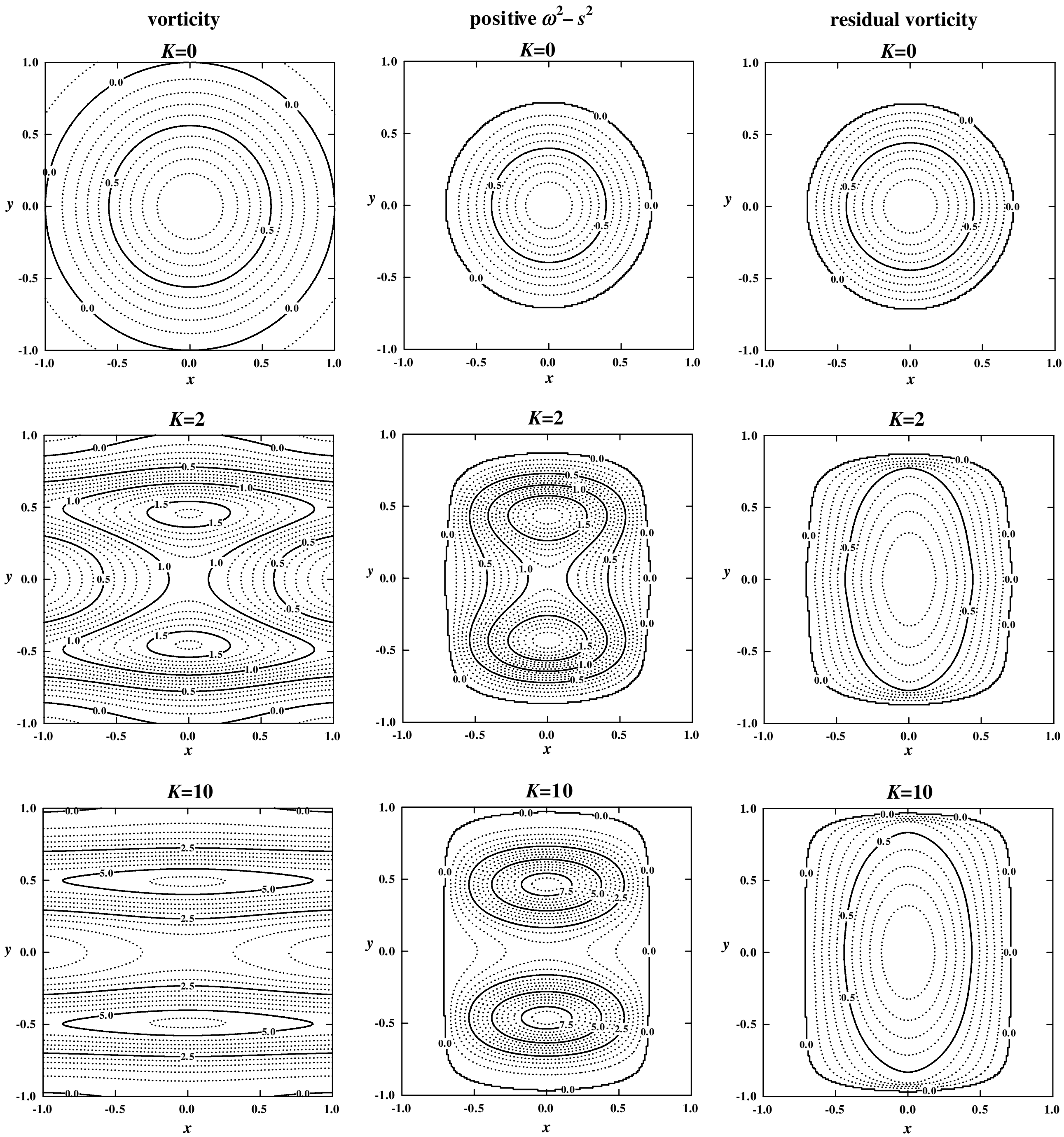

Fig. 5. Description of a planar vortex-shear interaction in terms of different vortex characteristics.

a vortex due to the strong inherent bias towards $[|\omega| + |s|]$ expressed by

$$\begin{aligned}\omega^2 - s^2 &= (\operatorname{sgn}\omega)[|\omega| - |s|] \cdot (\operatorname{sgn}\omega)[|\omega| + |s|] \\ &= \omega_{\mathrm{RES}} \cdot (\omega + \omega_{\mathrm{SH}}) \quad \text{for } |s| < |\omega|. \qquad (30)\end{aligned}$$

The above mentioned bias can be inferred from the geometry of the local velocity field near a point depicted in the BRF. The vortex intensity described in terms of $\omega^2 - s^2$ or, alternatively, ω_{RES} is examined in Fig. 6. Either of these measures requires specific conditions which guarantee the same local intensity of the swirling motion at different points of the flow field. Fig. 6 implies that a superimposed linear shearing onto the rigid-body rotation near a point (i.e. let $L_1 > L_2$, L_1 is otherwise arbitrary, L_2 is fixed) clearly affects (makes greater) the intensity of the swirling motion measured by $\omega^2 - s^2$ while ω_{RES} remains unaffected.

The criteria given by positive $\omega^2 - s^2$ and non-zero ω_{RES} are equivalent only at zero threshold as these criterial

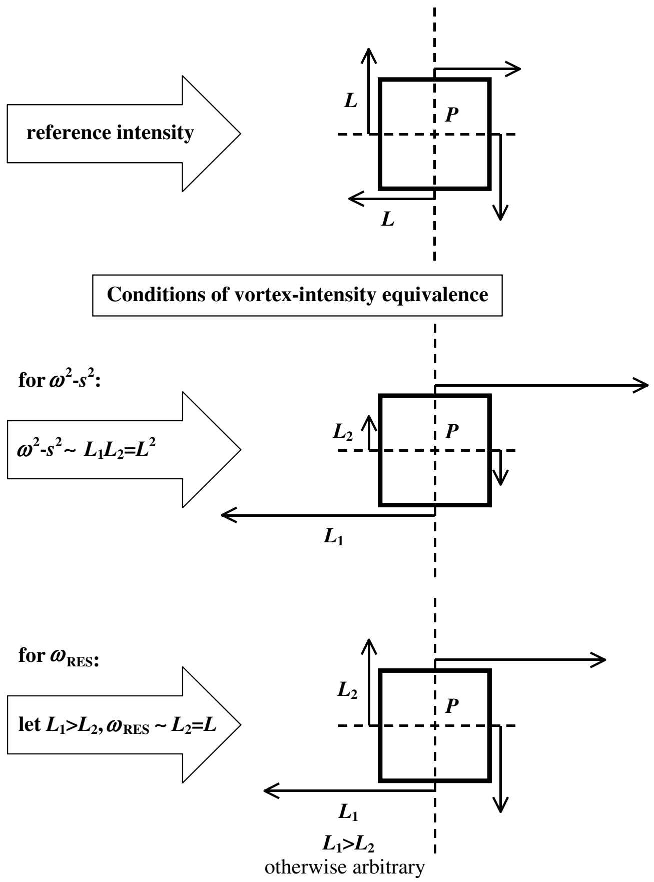

Fig. 6. Conditions of vortex-intensity equivalence: the comparison of $\omega^2 - s^2$ with ω_{RES}.

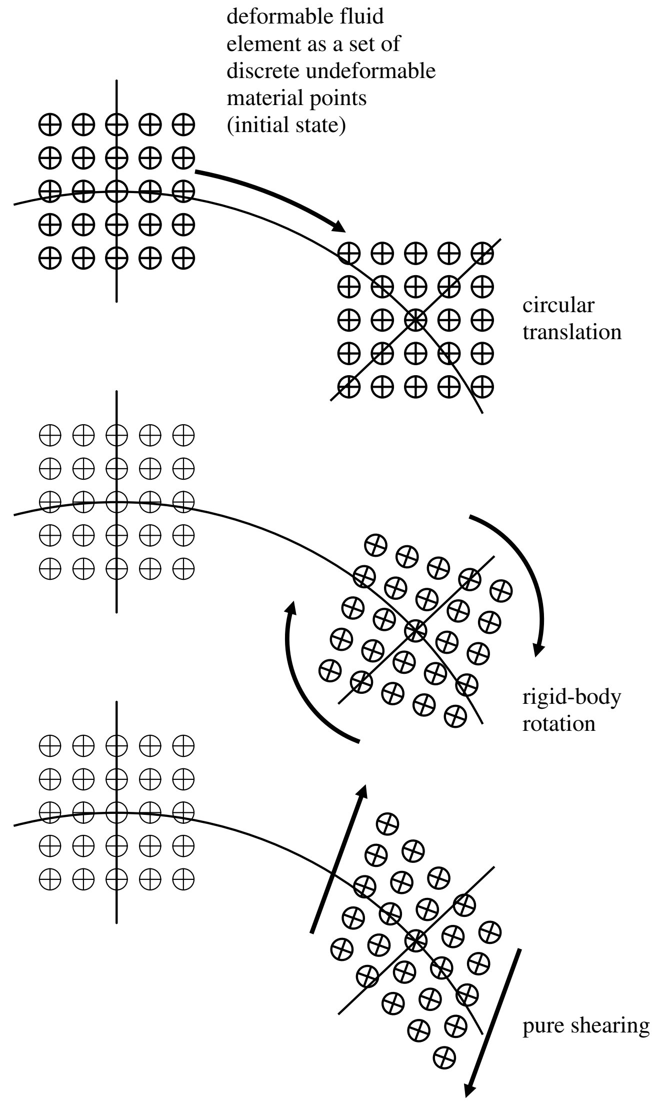

Fig. 7. Virtual motional changes of the fluid element within an ideal axisymmetric vortex (each at the final stage).

measures strongly differ in describing the inner structure of a vortex. The expression (30) explains the coincidence of the zero contour for $\omega^2 - s^2$ with that for ω_{RES} (Fig. 5). In both cases, this contour represents the boundary of a vortex. The boundary is, however, only roughly matching the original axisymmetric boundary of an isolated Taylor vortex. The reason is that any *local* schemes based on pointwise analysis can hardly reveal any virtual *non-local* flow components or features exactly.

The TDM, as the local concept, cannot reveal the global motion of an ideal axisymmetric vortex in concentric shearing layers. However, consistently with the TDM, the motion of an ideal vortex can be interpreted as the virtual superposition shown in Fig. 7 (for clarity including a circular translation) where the *residual* vorticity and *shear* vorticity apparently have the same signs. Hence, equally as in the case of Fig. 2, the superimposing construction in Fig. 7 is of a *non-destructive* nature and is applicable to infinitesimal motional changes only.

Finally, let us have a look at a primary interpretation of the *residual* vorticity. In the frame of the TDM, the *residual* vorticity can be interpreted in terms of (twice) the angular velocity of material points (Figs. 1 and 7) due to the *residual* rigid-body rotation. Now let us try to formulate a more conventional interpretation, for simplicity in 2D. Vorticity, perpendicular to a flow plane through a given point, is (twice) the *mean* angular velocity of any two instantaneously mutually orthogonal line segments, within the flow plane, going through the given point and, consequently, it is (twice) the *mean* angular velocity of all line segments, within the flow plane, going through the given point (this result is attributed to Cauchy, 1841, according to Truesdell and Toupin, 1960; Truesdell, 1954; Green, 1995). Then the *residual* vorticity in 2D can be interpreted in terms of (twice) the *least-absolute-value* angular velocity of all line segments, within the flow plane, going through the given point. Fig. 3 may help in this regard by considering dashed lines as the line segments with the minimum and maximum angular velocities. The *residual* vorticity is clearly zero for all the planar cases where strain rate dominates over vorticity as one portion of line segments rotates in one direction while the other portion of line segments in the opposite sense and two separating non-rotating line segments exist (saddle separatrices in Fig. 3). Further, for a simple shear one non-rotating line segment exists. The *shear* vorticity represents just the difference between (twice) the *mean* angular velocity and (twice) the *least-absolute-value* angular velocity of all line segments. In 3D, the already stated quasi-conventional interpretation of both the *residual* vorticity and the *shear* one is for equal leading-diagonal elements applicable to the BRF coordinate directions and planes, however, not to arbitrarily chosen directions and planes. Such universality is possessed by the conventional interpretation of vorticity, strictly said, the vorticity projection onto the normal of the examined flow plane. The above introduced interpretation of the *residual* vorticity is obviously a good argument for using this measure in vortex identification.

6. Conclusions

A brief survey of popular vortex-identification methods is presented on the background of other vortex-identification schemes and relevant discussions on vortex definition. A particular emphasis is put on the summary of new general requirements and limitations for any vortex-identification schemes and their underlying criterial quantities.

The present paper further suggests that a specific portion of vorticity provides a proper physical quantity for the *kinematic* identification of a vortex. On the basis of the triple decomposition of the relative motion near a point, the vorticity is decomposed into two parts, *shear* vorticity and *residual* vorticity. The latter is associated with the local *residual* rigid-body rotation near a point obtained after the extraction of an *effective* pure shearing motion and represents a direct measure of the actual swirling motion of a vortex. The *residual* vorticity retains all the very useful vorticity features and satisfies most of the general requirements for vortex identification.

Acknowledgements

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Appendix A

The transformation matrix **Q** for an arbitrarily rotated coordinate frame can be obtained by a sequence of three rotational (positive counterclockwise) transformations while the Cartesian coordinate system changes three times, as follows:

First rotation is around the z-axis by an angle α, $0 \leqslant \alpha \leqslant \pi$, $(x,y,z) \to (x^*,y^*,z^*)$, $z^* \equiv z$,

$$\mathbf{Q} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

Second rotation is around the y^*-axis by an angle β, $0 \leqslant \beta \leqslant \pi$, $(x^*,y^*,z^*) \to (x^{**},y^{**},z^{**})$, $y^{**} \equiv y^*$,

$$\mathbf{Q} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}. \tag{A.2}$$

Third rotation is around the z^{**}-axis by an angle γ, $0 \leqslant \gamma \leqslant \pi/2$, $(x^{**},y^{**},z^{**}) \to (x^{**},y^{**},z^{**})$, $z^{**} \equiv z^{**}$,

$$\mathbf{Q} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\beta\cos\gamma \\ -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}. \tag{A.3}$$

An arbitrary vector **v** in old coordinates is given by **Qv** in new coordinates, an arbitrary tensor **G** in old coordinates is described by $\mathbf{QGQ}^{\mathrm{T}}$ in new coordinates.

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